REFERENCES

[1] Carlin, H. J., Cascade transmission line synthesis, Rept PIBMRI-889-61, Microwave Research Inst., Polytechnic Institute of Brooklyn, N. Y., Apr 1961.
[2] Richards, P. I., Resistor-transmission-line circuits, *Proc. IRE*, vol 37, Feb 1948, pp 217–220.
[3] Collin, R. E., Theory and design of wide-band multisection quarter-wave transformers, *Proc. IRE*, vol 43, Feb 1955, pp 179–185.
[4] Kasahara, Y., H. Ozaki, and T. Fujisawa, Design of distributed constant filter, Osaka Univ. Tech. Rept., vol 4, no 115, 1954.
[5] Ozaki, H., and J. Ishii, Synthesis of transmission-line networks and the designs of UHF filters, *IRE Trans. on Circuit Theory*, vol CT-2, Dec 1955, pp 325–336.
[6] Ikeno, N., The design theory of networks with distributed constants, *Rept. of Res. and Dev.*, (Japan Tel. and Tel. Co.), vol 4, Jul 1955, pp 379–417.
[7] ——, Design and distributed constant networks, *Elec. Commun. Lab., Tech. J.*, vol 4, May 1956, pp 29–36.
[8] Jones, E. M. T., Synthesis of wide-band microwave filters to this prescribed insertion loss, *1956 IRE Conv. Rec.*, pt V, pp 119–126.
[9] Riblett, H. J., General synthesis of quater-wave impedance transformers, *IRE Trans. on Microwave Theory and Techniques*, vol MTT-5, Jan 1957, pp 36–43.
[10] Young, L., Stepped-impedance transformers and filter prototypes, *IRE Trans. on Microwave Theory and Techniques*, vol MTT-10, Sep 1962, pp 339–359.
[11] Wenzel, R. J., Exact design of TEM microwave networks using quater-wave lines, *IRE Trans. on Microwave Theory and Techniques*, vol MTT-12, Jan 1964, pp 94–111.
[12] Sharpe, C. B., A general Tchebycheff, rational function, *Proc. IRE*, vol 42, Feb 1954, pp 454–457.
[13] Helman, D., Tchebycheff approximations for amplitude and delay with rational functions, *Proc. of Symp. on Modern Network Synthesis*, Polytechnic Institute of Brooklyn, N. Y., vol 5, Apr 1955, pp. 385–402.
[14] Carlin, H. J., and W. Kohler, Network theory with transmission line elements, Rept PIBMRI-1046-62, Microwave Research Inst., Polytechnic Institute of Brooklyn, N. Y., Jun 25, 1962.
[15] The functions discussed in the present paper and in [14] are presented from a somewhat different point of view in Riblett, H. J., The application of a new class of equal-ripple functions to some familiar transmission-line problems, *IEEE Trans. on Microwave Theory and Techniques*, vol MTT-12, Jul 1964, pp 415–421.
[16] Carlin, H. J., and A. B. Giordano, *Network Theory*. Englewood Cliffs, N. J.: Prentice-Hall, Jun 1964, ch 4.
[17] Bode, H. W., *Network Analysis and Feedback Amplifier Design*. Princeton, N. J.: Van Nostrand, 1945, pp 363–367.
[18] Fano, R., Theoretical limitations on broadband matching of arbitrary impedances, *J. Franklin Inst.*, vol 249, Jan–Feb 1950, pp 57–98, and pp 139–154, (see Tables I, II, and III, pp 68–69).

Transverse Magnetic Wave Propagation in Sinusoidally Stratified Dielectric Media

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***Abstract*—The problem of the propagation of TM waves in a sinusoidally stratified dielectric medium is considered. The propagation characteristics are determined from the stability diagram of the resultant Hill's equation. Numerical results show that the stability diagrams for Hill's equation and those for Mathieu's equation are quite different. Consequently, the dispersion properties of TM waves and TE waves in this stratified medium are also different. Detailed dispersion characteristics of TM waves in an infinite stratified medium and in waveguides filled longitudinally with this stratified material are obtained.**

INTRODUCTION

THE PROBLEM OF electromagnetic wave propagation in a sinusoidally varying dielectric medium is not only of interest from a theoretical point of view but also possesses many possible applications [1], [2]. For example, a section of waveguide filled with this type of inhomogeneous dielectric may be used as a bandpass filter in the mm or in the optical range. The use of an ultrasonic standing wave as a modulating device for certain pressure sensitive media, such as carbon disulfide, pentane, or nitric acid at optical frequencies to achieve a sinusoidally varying dielectric medium may be proposed. Other applications, such as the study of acoustically modulated plasma column and the analysis of sinusoidally modulated dielectric slab antenna, have also been proposed. Furthermore, the results should be very useful in the study of wave propagation in solids [3].

It can be shown [2] that two types of waves, propagating in the direction of the dielectric inhomogeneity, may exist: one with its electric vector transverse to the direction of propagation called a TE wave, and the other with its magnetic vector transverse to the direction of propagation, called a TM wave. The resultant differential equations for TE waves and TM waves are, respectively, the Mathieu and the Hill differential equations [4], [5]. The simpler case of the propagation of TE waves in a sinusoidally stratified dielectric medium has been considered most recently by Tamir, Wang and Oliner [1], and discussed briefly by Yeh and Kaprielian [2]. The purpose of this paper is to consider the problem of the propagation of TM waves in such an inhomogeneous medium. Since the solution of a Hill equation is required, it is expected that the results will be rather in-

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volved. It is found that the propagation characteristics of TM waves are quite different from those of TE waves for large dielectric variations.

Formulation of the Problem

It is assumed that the inhomogeneous dielectric medium under consideration fills the entire space and possesses a relative permittivity

$$\frac{\epsilon(z)}{\epsilon_0} = A\left(1 - \delta \cos\frac{2\pi z}{d}\right) \tag{1}$$

and a relative permeability

$$\frac{\mu}{\mu_0} = 1 \tag{2}$$

in the (x, y, z) rectangular coordinates. ϵ_0 and μ_0 are, respectively, the free space permittivity and permeability. A and δ are known positive constants. Furthermore, $0 \leq \delta < 1$. d denotes the period of the sinusoidal variation (Fig. 1).

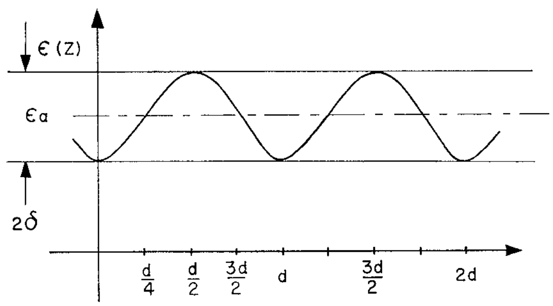

Fig. 1. Variation of permittivity as a function of longitudinal distance.

The source-free vector wave equations in this medium are:

$$\nabla \times \nabla \times \boldsymbol{E} - k_0^2(\epsilon(z)/\epsilon_0)\boldsymbol{E} = 0 \tag{3}$$

$$\nabla \times \nabla \times \boldsymbol{H} - \frac{\nabla\epsilon(z)}{\epsilon(z)} \times \nabla \times \boldsymbol{H} - k_0^2(\epsilon(z)/\epsilon_0)\boldsymbol{H} = 0 \tag{4}$$

where $\boldsymbol{E}$ and $\boldsymbol{H}$ are, respectively, the electric and magnetic field vectors, $k_0^2 = \omega^2\mu_0\epsilon_0$, and a time dependence $e^{-i\omega t}$ is assumed. It can be shown that all field components in this medium can be obtained from the scalar quantities $\Phi(x, y, z)$ and $\Psi(x, y, z)$ as follows [2]:

$$\boldsymbol{E}^{(m)} = \nabla \times [\Phi(x, y, z)\boldsymbol{e}_z] \tag{5}$$

$$\boldsymbol{H}^{(m)} = \frac{-i}{\omega\mu_0} \nabla \times \nabla \times [\Phi(x, y, z)\boldsymbol{e}_z] \tag{6}$$

for transverse electric waves; and

$$\boldsymbol{H}^{(e)} = \nabla \times [\Psi(x, y, z)\boldsymbol{e}_z] \tag{7}$$

$$\boldsymbol{E}^{(e)} = \frac{i}{\omega\epsilon(z)} \nabla \times \nabla \times [\Psi(x, y, z)\boldsymbol{e}_z] \tag{8}$$

for transverse magnetic waves. $\boldsymbol{e}_z$ is the unit vector in the z direction. Upon substituting (5) into (3), and (7) into (4), carrying out the vector operations and separating variables in rectangular coordinates, one obtains

$$\Phi(x, y, z) = \begin{Bmatrix}\sin \\ \cos\end{Bmatrix}(sx) \begin{Bmatrix}\sin \\ \cos\end{Bmatrix}(wy) \left\{U^{(1),(2)}(z)\right\} \tag{9}$$

and

$$\Psi(x, y, z) = \begin{Bmatrix}\sin \\ \cos\end{Bmatrix}(px) \begin{Bmatrix}\sin \\ \cos\end{Bmatrix}(qy) \left\{V^{(1),(2)}(z)\right\} \tag{10}$$

where s, w, p, and q are separation constants. $U^{(1),(2)}(z)$ and $V^{(1),(2)}(z)$ satisfy, respectively, the differential equations

$$\left\{\frac{d^2}{dz^2} + \left[k_0^2(\epsilon(z)/\epsilon_0) - s^2 - w^2\right]\right\} U^{(1),(2)}(z) = 0 \tag{11}$$

and

$$\left\{\frac{d^2}{dz^2} - \left(\frac{d\epsilon(z)}{dz}\right)\frac{1}{\epsilon(z)}\frac{d}{dz} + \left[k_0^2(\epsilon(z)/\epsilon_0) - p^2 - q^2\right]\right\} V^{(1),(2)}(z) = 0. \tag{12}$$

Since we are only concerned with the propagation of transverse magnetic waves in this inhomogeneous medium, the transverse electric waves will not be considered further. Putting (1) into (12), introducing the dimensionless variable $\xi = \pi z/d$, and making the substitution

$$V^{(1),(2)}(z) = (1 - \delta \cos 2\xi)^{1/2} W^{(1),(2)}(\xi) \tag{13}$$

gives

$$\left[\frac{d^2}{d\xi^2} + \lambda(\xi)\right] W^{(1),(2)}(\xi) = 0 \tag{14}$$

where

$$\lambda(\xi) = \frac{2\delta \cos 2\xi}{1 - \delta \cos 2\xi} - \frac{3\delta^2 \sin^2 2\xi}{(1 - \delta \cos 2\xi)^2} + \left(\frac{k_0 d}{\pi}\right)^2 \cdot \left\{A - A\delta \cos 2\xi - \left[\left(\frac{p}{k_0}\right)^2 + \left(\frac{q}{k_0}\right)^2\right]\right\}. \tag{15}$$

It is noted that since $\lambda(\xi)$ is an even periodic function, it may be expanded in the Fourier cosine series

$$\lambda(\xi) = \theta_0 + 2\sum_{n=1}^{\infty} \theta_n \cos 2n\xi \tag{16}$$

in which

$$\theta_0 = \left(\frac{k_0 d}{\pi}\right)^2 \left[A - \left(\frac{p}{k_0}\right)^2 - \left(\frac{q}{k_0}\right)^2\right] - \left[\frac{1}{\sqrt{1 - \delta^2}} - 1\right] \tag{17a}$$

$$\theta_1 = -\frac{\delta}{2}\left(\frac{k_0 d}{\pi}\right)^2 A + \frac{4b^3 - 2b}{b^2 - 1} \tag{17b}$$

$$\theta_n = \frac{(3n + 1)b^{n+2} - (3n - 1)b^n}{b^2 - 1} \quad (n \geq 2) \tag{17c}$$

with

$$b = \frac{1}{\delta} - \frac{1}{\delta}\sqrt{1-\delta^2}. \qquad (17d)$$

The above series converges absolutely for $0 \leq \delta < 1$. Substituting (16) into (14), one obtains

$$\left[\frac{d^2}{d\xi^2} + \theta_0 + 2\sum_{n=1}^{\infty} \theta_n \cos 2n\xi\right] W^{(1),(2)}(\xi) = 0 \qquad (18)$$

which is the general form of Hill's equation [4]-[6]. It is known that two types of solutions for the Hill equation exist: one called the stable type, and the other called the unstable type. In order to have propagating waves in the z direction, only the stable type is allowed.

Solutions of Hill's Equation

With the help of Floquet's Theorem [6] concerning wave propagation in periodic media, the solutions of Hill's equation can be expressed in the following form:

$$W^{(1),(2)}(\xi) = e^{\pm i\beta\xi} \sum_{n=-\infty}^{\infty} C_n(\beta) e^{\pm 2in\xi} \qquad (19)$$

where β and $C_n(\beta)$ are yet unknown coefficients. After substituting (19) into (18), and simplifying, one obtains the following recursion relations:

$$-(\beta + 2n)^2 C_n + \sum_{m=-\infty}^{\infty} \theta_m C_{n-m} = 0 \qquad (20)$$

$$(n = \cdots -2, -1, 0, 1, 2, \cdots)$$

with $\theta_{-m} = \theta_m$. The above is a set of an infinite number of homogeneous linear algebraic equations in C_n. For a nontrivial solution to exist the determinant of these equations must vanish. This equation is called the characteristic equation of the Hill equation. Using an ingenious method described in Morse and Feshbach [6], it is possible to simplify this characteristic equation to give

$$\sin^2 \frac{\pi\beta}{2} = \Delta(0) \sin^2 \frac{\pi\sqrt{\theta_0}}{2} \qquad (21)$$

where $\Delta(0)$ is the determinant of the matrix $[M]$, whose elements are

$$M_{mm} = 1$$

$$M_{mn} = \frac{-\theta_{m-n}}{4m^2 - \theta_0} \qquad m \neq n. \qquad (22)$$

The characteristic number β can be obtained from (21).

Real values of β yield stable solutions to Hill's equation, while complex values of β produce unstable solutions. Physically speaking the stable solutions correspond to modulated propagating waves, and the unstable solutions correspond to damped or growing waves. (The fields for the growing waves do not satisfy the radiation condition at infinity, hence they must be omitted.)

Numerical computation has been carried out for (21). The values for the infinite determinant $\Delta(0)$ were obtained by the successive approximation method [7]. In other words, computations were carried out for a 3×3 determinant, a 4×4 determinant, a 5×5 determinant, etc., until the desired accuracy was reached. It was found (numerically) that the infinite determinant converges quite rapidly within the present region of interests. At no time was any determinant greater than 7×7 required to achieve an accuracy of three significant figures.

Results of the computation are given in terms of a "stability diagram," which is customary in the study of Hill-type equations. Figures 2 and 3 show, respectively, the "stability diagram" for the cases $\delta = 0.25$ and $\delta = 0.4$. The unshaded areas in these figures are the "stable regions" wherein β is purely real; the shaded areas are the "unstable regions" wherein β is complex. It is noted that the value of β in the unshaded regions is bounded by

$$m \leq |\beta| \leq m + 1 \quad (m = 0, 1, 2, \cdots) \qquad (23)$$

so that the value of m may be used to label the appropriate regions as shown in Figs. 2 and 3.

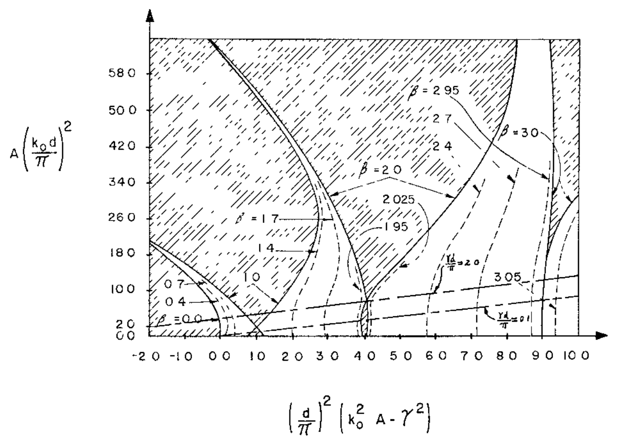

Fig. 2. Stability chart for Hill's equation with $\delta = 0.25$. Unstable regions are shaded. Family of straight lines represents (28) for various values of $\gamma d/\pi$.

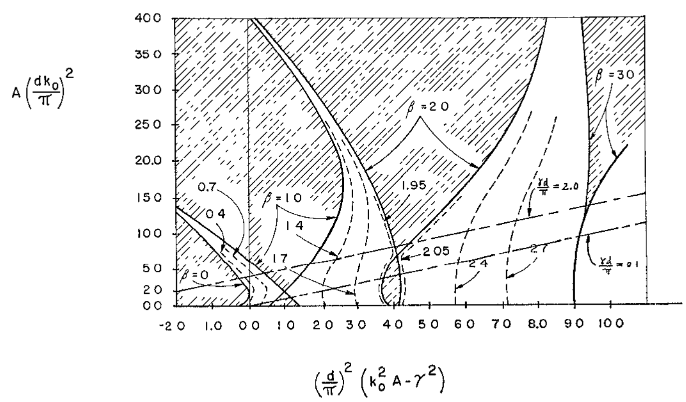

Fig. 3. Stability chart for Hill's equation with $\delta = 0.4$. Unstable regions are shaded. Family of straight lines represent (28) for various values of $\gamma d/\pi$.

It is interesting to note the differences between the stability diagram for Mathieu equation and those given in Figs. 2 and 3. Unlike the Mathieu case, curves separating the stable and unstable regions do not necessarily meet at the abscissa. As a matter of fact, in some instances they cross over each other (as can be seen from these figures) near the point $\theta_0=4.0$. As δ increases the overlapped region becomes larger.

Propagation Characteristics of TM Waves

A. Infinite region filled with sinusoidally stratified dielectric

The transverse magnetic field components of a TM wave in an infinite medium filled with sinusoidally stratified dielectric can be obtained from (7) and (10):

$$H_x^{(e)} = \sum_{n=-\infty}^{\infty} iqC_n e^{ipx} e^{iqy} e^{i(\beta+2n)\pi z/d}\left(1-\delta\cos\frac{2\pi z}{d}\right)^{1/2} \quad (24)$$

$$H_y^{(e)} = \sum_{n=-\infty}^{\infty} -ipC_n e^{ipx} e^{iqy} e^{i(\beta+2n)\pi z/d} \cdot\left(1-\delta\cos\frac{2\pi z}{d}\right)^{1/2} \quad (25)$$

where the coefficients C_n can be determined from (20) in terms of C_0. C_0 is obtained from a normalization condition. All electric field components may be found from Maxwell's equations.

Unlike the case of a TM wave propagating in an infinite homogeneous medium in which β is simply related to p and q by the following:

$$\beta^2 = k^2 - \gamma^2$$

with $\gamma^2=p^2+q^2$ and $k^2=\omega^2\mu\epsilon$ where μ and ϵ are the permeability and permittivity of the homogeneous medium, the propagation constant β for the inhomogeneous case is related to p and q through the stability diagrams given by Figs. 2 and 3. Real values of β as a function of real values of γ for a fixed value of A, δ, and k_0d are shown in Figs. 4 through 6. It is recalled that complex values of β indicate the presence of damped waves (i.e., nonpropagating waves). p and q are taken to be real. The unshaded regions in these figures indicate the regions in which β is real (i.e., regions in which propagating waves may exist). One notes from these figures that for very small values of k_0d, say $k_0d<0.2$, as long as $\gamma^2<k_0^2A$, β is always real. However as k_0d increases, there exist regions in which β is complex even though $\gamma^2<k_0^2A$. The presence of these stop band and pass band regions is characteristic of wave propagation in periodic structures [3].

B. Waveguide filled longitudinally with sinusoidally stratified dielectric

It is assumed that a rectangular waveguide of dimension h_1 and h_2 is filled completely with an inhomogeneous dielectric medium, which varies sinusoidally in the longitudinal direction. The general expressions for the transverse magnetic field components of a TM wave are:

$$H_x^{(e)} = \sum_{m=1}^{\infty}\sum_{r=1}^{\infty}\sum_{n=-\infty}^{\infty} C_n^{m,r}\sin\frac{m\pi x}{h_1}\cos\frac{r\pi y}{h_2}e^{i(\beta^{m,r}+2n)\pi z/d} \cdot\frac{r\pi}{h_2}\left(1-\delta\cos\frac{2\pi z}{d}\right)^{1/2} \quad (26)$$

$$H_y^{(e)} = \sum_{m=1}^{\infty}\sum_{r=1}^{\infty}\sum_{n=-\infty}^{\infty} C_n^{m,r}\cos\frac{m\pi x}{h_1}\sin\frac{r\pi y}{h_2}e^{i(\beta^{m,r}+2n)\pi z/d} \cdot\left[-\left(\frac{m\pi}{h_1}\right)\right]\left(1-\delta\cos\frac{2\pi z}{d}\right)^{1/2} \quad (27)$$

where $C_0^{m,r}$ are arbitrary constants, and $C_n^{m,r}$ can be determined from (20) in terms of $C_0^{m,r}$. Expressions for the electric field components can easily be derived from Maxwell's equations.

To obtain the dispersion curves for this case, one combines (17a) and (17b)

$$\theta_0+\frac{2\theta_1}{\delta} = -\left(\frac{\gamma d}{\pi}\right)^2 - \left[\frac{1}{\sqrt{1-\delta^2}}-1\right] + \frac{4b^3-2b}{b^2-1} \quad (28)$$

where b is given by (17d) and

$$\gamma^2 = \left(\frac{m\pi}{h_1}\right)^2 + \left(\frac{r\pi}{h_2}\right)^2. \quad (29)$$

Expression (28) is the equation of a family of straight lines as shown in Figs. 2 and 3. For given values of $\gamma d/\pi$ and δ one can then obtain the $\omega-\beta$ diagram from these figures. The $\omega-\beta$ diagrams are given for $\gamma d/\pi = 0.1$, 2.0, 5.0, and $\delta=0.25$, 0.40 in Figs. 7 through 12. The pass band and stop band characteristics can clearly be seen. It is interesting to note that the first pass band starts at a frequency which is lower than the cutoff frequency for an identical waveguide filled with homogeneous dielectric material, which has a dielectric constant equal to the average value of the inhomogeneous dielectric.

The $\omega-\beta$ diagrams given by Figs. 7 through 12 are also applicable for circular waveguide except γ^2 is given by $(\Gamma_{mr}/\rho_0)^2$, where ρ_0 is the radius of the circular waveguide and Γ_{mr} are the roots of the equation

$$J_m(\Gamma_{mr}) = 0.$$

References

[1] Tamir, T., H. C. Wang, and A. A. Oliner, Wave propagation in sinusoidally stratified dielectric media, *IEEE Trans. on Microwave Theory and Techniques*, vol MTT-12, May 1964, pp 323–335.

[2] Yeh, C., and Z. A. Kaprielian, On inhomogeneously filled waveguides, USCEC Rept 84-206, Electrical Engineering Dept., University of Southern California, Los Angeles, 1963.

[3] Brillouin, L., *Wave Propagation in Periodic Structures*. New York: Dover, 1953.

[4] McLachlan, N. W., *Theory and Application of Mathieu Functions*. Oxford, England: University Press, 1951.

[5] Meixner, J., and F. W. Schäfke, *Mathieusche Funktionen und Sphäroidfunktionen*. Berlin, Germany: Springer-Verlag, 1954.

[6] Morse, P. M., and H. Feshbach, *Methods of Theoretical Physics*. New York: McGraw-Hill, 1953.

[7] Kantorovich, L., and V. Krylov, *Approximate Methods of Higher Analysis*. New York: Interscience Publishers, Inc., 1958.

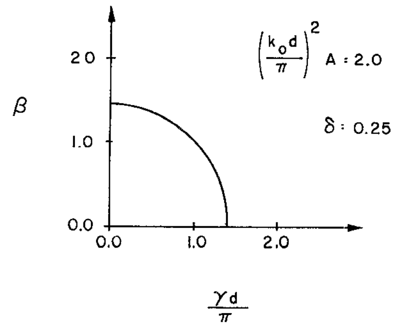

Fig. 4. The propagation constant β as a function of $\gamma d/\pi$.

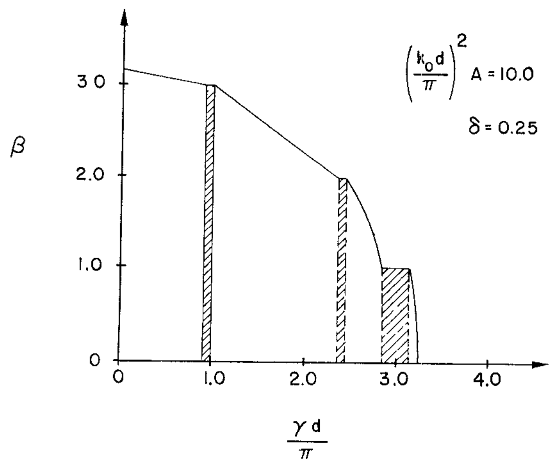

Fig. 5. The propagation constant β as a function of $\gamma d/\pi$. Unstable regions are shaded.

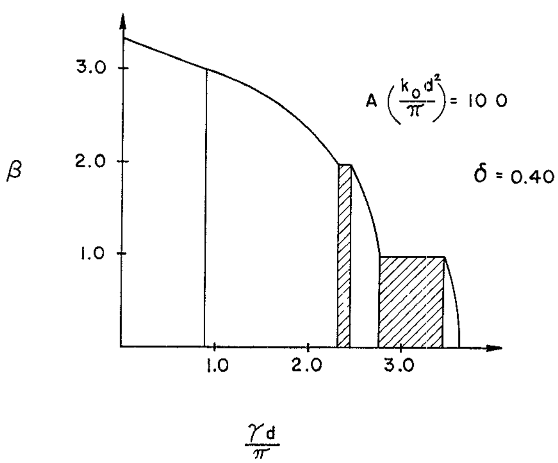

Fig. 6. The propagation constant β as a function of $\gamma d/\pi$. Unstable regions are shaded.

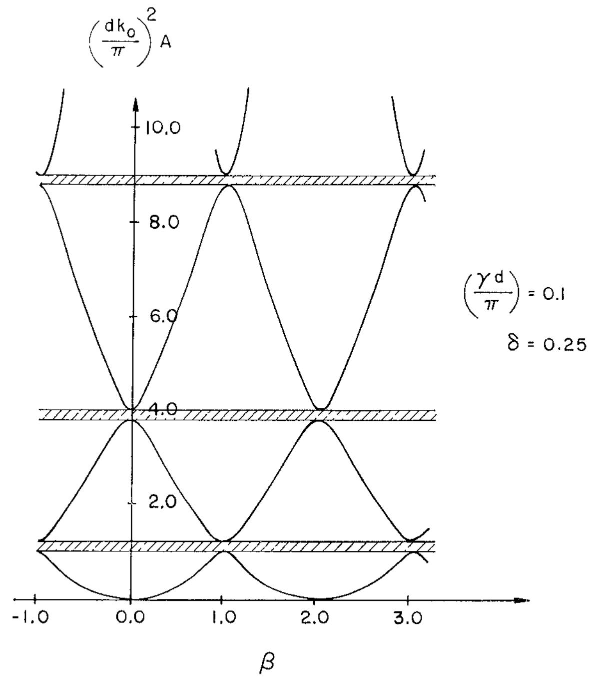

Fig. 7. Frequency as a function of β with $\delta=0.25$. Stop bands are shaded.

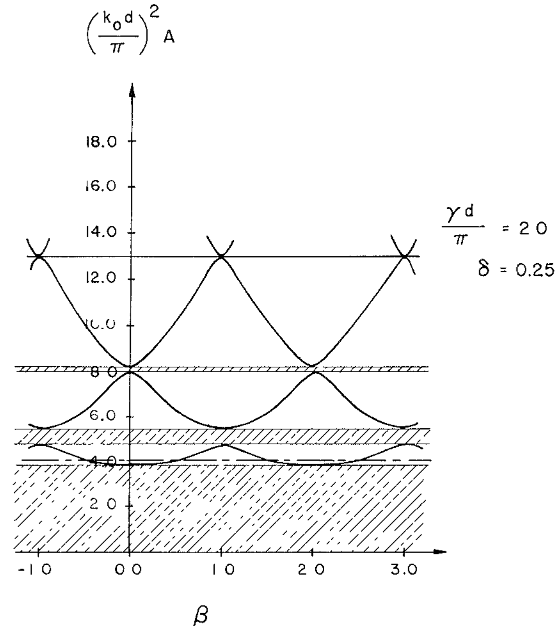

Fig. 8. Frequency as a function of β with $\delta=0.25$. Stop bands are shaded.

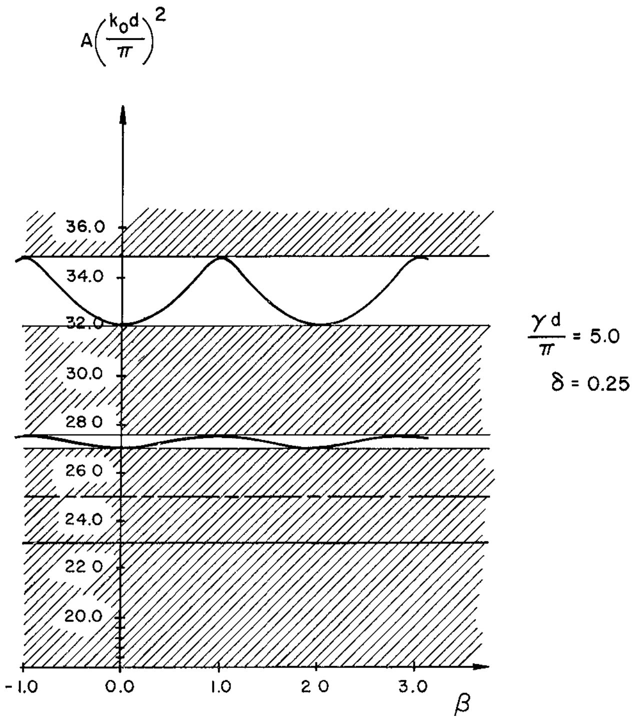

Fig. 9. Frequency as a function of β with $\delta=0.25$. Stop bands are shaded. The dot-dash line represents the cutoff frequency of a waveguide filled with a homogeneous dielectric medium with $\epsilon=\epsilon_a$. A very narrow stable region near $A(k_0d/\pi)^2=23.0$ is indicated by a solid line.

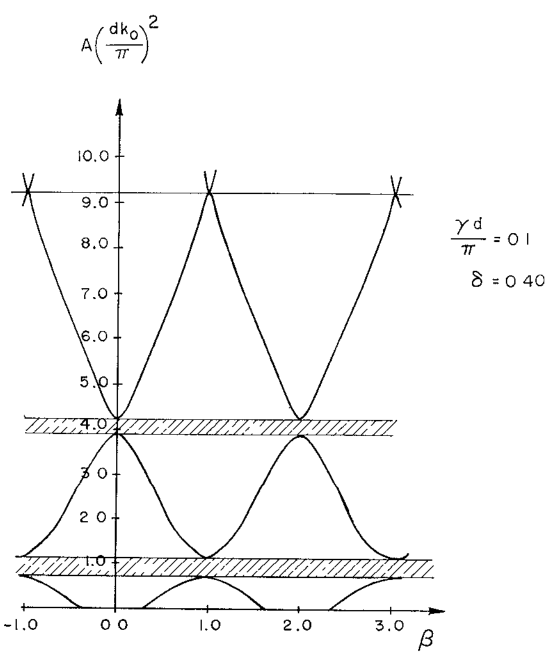

Fig. 10. Frequency as a function of β with $\delta=0.4$. Stop bands are shaded.

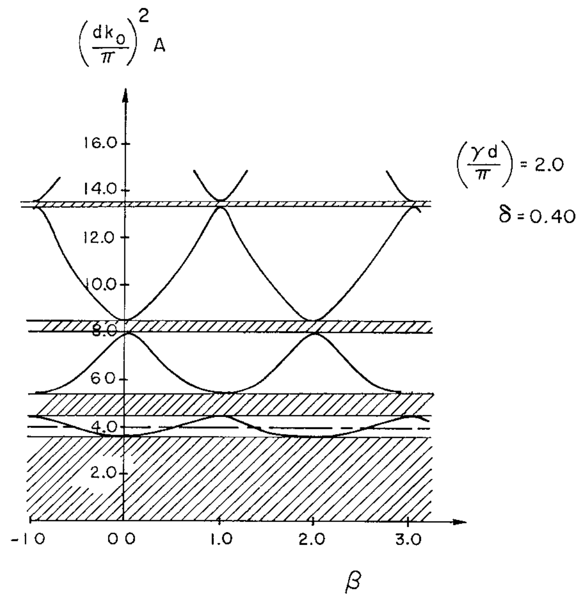

Fig. 11. Frequency as a function of β with $\delta=0.4$. Stop bands are shaded. The dot-dash line represents the cutoff frequency of a waveguide filled with a homogeneous dielectric medium with $\epsilon=\epsilon_a$.

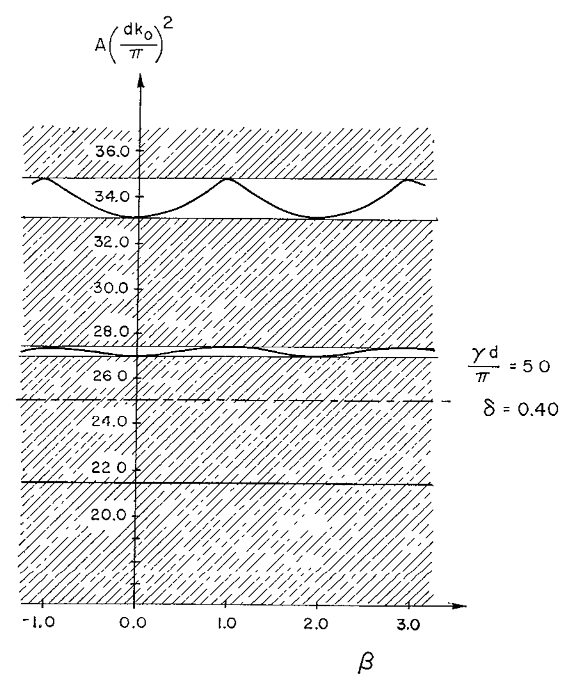

Fig. 12. Frequency as a function of β with $\delta=0.4$. Stop bands are shaded. The dot-dash line represents the cutoff frequency of a waveguide filled with a homogeneous dielectric medium with $\epsilon=\epsilon_a$. A very narrow stable region near $A(dk_0/\pi)^2=21$ is indicated by a solid line.